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D_n Quivers From Branes

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Abstract: D-branes can end on orbifold planes if the action of the orbifold group includes $(-1)^{F_L}$. We consider configurations of D-branes ending on such orbifolds and study the low-energy theory on their worldvolume. We apply our results to gauge theories with eight supercharges in three and four dimensions. We explain how mirror symmetry for $N = 4$ $d = 3$ gauge theories with gauge group $Sp(k)$ and matter in the antisymmetric tensor and fundamental representations follows from S-duality of IIB string theory. We argue that some of these theories have hidden Fayet-Iliopoulos deformations, not visible classically. We also study a class of finite $N = 2$ $d = 4$ theories (so-called D_n quiver theories) and find their exact solution. The integrable model corresponding to the exact solution is a Hitchin system on an orbifold Riemann surface. We also give a simple derivation of the S-duality group of these theories based on their relationship to $SO(2n)$ instantons on $\mathbf{R}^2 \times \mathbf{T}^2$.

Keywords: Field Theories in Lower Dimensions, D-branes, Supersymmetry and Duality, Extended Supersymmetry.

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Contents

1. Introduction

Constructing field theories from brane configurations in string theory can be of great help in understanding the properties of the former (see refs. [1, 2, 3] for some examples and ref. [4] for a review.) In this letter we use this approach to investigate certain aspects of gauge theories with eight supercharges in three and four dimensions. One of our results is a simple proof of mirror symmetry for $N = 4$ $d = 3$ theories with gauge group $Sp(k)$ and matter in the antisymmetric tensor and fundamental representations [5, 6]. Our argument is based on S-duality of IIB string theory and is completely analogous to that for unitary groups [1, 6]. (A different derivation of mirror symmetry covering both unitary and symplectic groups was presented in ref. [7].) We also argue that $Sp(k)$ gauge theories with an antisymmetric tensor and two or three flavors of fundamentals have accidental $U(1)$ symmetries in the infrared and therefore hidden Fayet-Iliopoulos (FI) deformations. (Classically, $Sp(k)$ gauge theories do not admit FI deformations.) Another result is an exact solution of a class of finite $N = 2$ $d = 4$ gauge theories, so-called D_n quiver theories [8]. Recall that quiver gauge theories are finite if and only if the quiver is an affine Dynkin diagram of type A, D, or E [8]. The exact solutions of these theories have been found in ref. [8] via geometric engineering. The A_n case can also be solved using M-theory [2] or compactification to three dimensions [9]. In this paper we extend the latter approach to D_n quivers. We explain how the exact solution of D_n quiver theories is related to the moduli space of D_n instantons on $\mathbf{R}^2 \times \mathbf{T}^2$. We use this correspondence to compute the S-duality group of the theories. The results are in agreement with the geometric engineering approach. Further, we show that the solution of the model is encoded in the moduli space of Hitchin equations on an orbifold Riemann surface and derive the corresponding Seiberg-Witten curve.

The brane configurations that we use are somewhat unusual: they involve D-branes ending on orbifold planes. Let us remind why it is possible for D-branes to end on certain orbifolds [10]. Consider an orientifold 5-plane with negative charge and a D5-brane on top of it. Such a configuration has sixteen unbroken supersymmetries. It also has a vanishing total Ramond-Ramond charge, and therefore a very simple S-dual: an orbifold $\mathbf{R}^6 \times \mathbf{R}^4/\mathcal{I}$, where $\mathcal{I}$ is a product of $(-1)^{F_L}$ and the inversion of all four coordinates of $\mathbf{R}^4$ [11]. (The S-dual of an $O5^-$ plane alone is an object which is magnetically charged with respect to the Neveu-Schwarz B-field. It does not have a simple conformal field theory description.) Since fundamental strings and D3-branes can end on D5-branes, S-duality of IIB string theory implies that D-strings and D3-branes must be able to end on such orbifold planes. The boundary states describing D-branes ending on orbifolds have been constructed in ref. [10]. To be more precise, the theory on the orbifold worldvolume is a $(1,1)$ $d=6$ theory with gauge group $SO(2)$. (The $SO(2)$ vector multiplet arises from the twisted closed string sector.) The end of a D3-brane is a 2-brane magnetically charged with respect to the $SO(2)$ gauge field. This charge can be positive or negative, so there are two possible boundary states describing a D3-brane ending on the orbifold. We will call them $|D3+\rangle$ and $|D3-\rangle$. Similarly, the end of a D-string is a 0-brane electrically charged with respect to the $SO(2)$.

What is the analogue of this in IIA string theory? There one starts with an $O4^-$ plane and asks about its strong coupling limit, i.e. about its lift to M-theory. One finds [12, 13] that the corresponding M-theory background is $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{R}^5/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts as an inversion on $\mathbf{R}^5$ and in addition flips the sign of the 3-form potential. Now we can go back to IIA string theory by compactifying one of the directions of $\mathbf{R}^5/\mathbf{Z}_2$. The resulting object is an orbifold 5-plane of some kind which is magnetically charged with respect to the B-field. Its charge is minus the charge of the NS5-brane [12, 13]. It is related to NS5-branes in the same way as an $O4^-$ plane is related to D4-branes, so one could call it a "Neveu-Schwarz orientifold." This object does not have a simple CFT description. However, if we put an NS5-brane on top of it, the total charge becomes zero. The conformal field theory description of this composite object is very simple: it is a IIA orbifold $\mathbf{R}^6 \times \mathbf{R}^4/\mathcal{I}$ [14, 11], where $\mathcal{I}$ is the same as before. It can be shown that the twisted closed string sector gives rise to a single $(2,0)$ tensor multiplet propagating on the orbifold plane [14]. This tensor multiplet describes small fluctuations of the NS5-brane sitting at the orbifold plane. Furthemore, since the composite orbifold contains an NS5-brane, a D4-brane must be able to end on it. The end of the D4-brane is a 3-brane magnetically charged with respect to the periodic scalar which is a part of the tensor multiplet. This charge can be of either sign.

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The organization of this paper is as follows. In section 2 we investigate the low-energy theory of D-branes ending on orbifolds. In section 3 we use D3-branes ending on orbifolds to study mirror symmetry in $d = 3$. In section 4 derive the Seiberg-Witten solution for D_n quiver theories in $d = 4$.

2. The brane configuration

Consider an orbifold 5-plane $\mathbf{R}^6 \times \mathbf{R}^4/\mathcal{I}$ extending in the $x^0, x^1, \ldots, x^5$ directions and localized at $x^6 = \cdots = x^9 = 0$. Let k D3-branes extend in the x^0, x^1, x^2, x^6 directions and end on the orbifold plane. Such a configuration preserves eight supersymmetries. This is true even if some D3-branes carry opposite $SO(2)$ magnetic charges [10]. This can be seen from the partition function for an open string with ends on a pair of such D3-branes. The partition function can be represented as

$$\int_0^\infty dl \ \langle D3+|e^{-lH_{cl}}|D3-\rangle \, , \tag{2.1}$$

where $|D3+\rangle$, $|D3-\rangle$ are boundary states describing D-branes and H_{cl} is the closed string Hamiltonian. This expression can be evaluated using the formulas in ref. [10]. The matrix element in eq. (2.1) does not exhibit a divergence for $l \to 0$, which would be a signal of tachyon instability. Thus D3-branes with opposite $SO(2)$ charge do not exert force on each other. Similarly, there is no net force between D3-branes with the same $SO(2)$ charge.

If not for the orbifold, the low-energy theory on the world-volume of D3-branes would be an $N = 4$ $U(k)$ gauge theory. The orbifold projection breaks supersymmetry down to $N = 2$. Furthemore, if k_+ D3-branes have $SO(2)$ charge $+1$, and k_- of them have $SO(2)$ charge -1 ($k = k_+ + k_-$), the projection breaks $U(k)$ down to $U(k_+) \times U(k_-)$. This can be seen from the fact that only D3-branes with the same $SO(2)$ charge can be regarded as indistinguishable, and therefore the Weyl group is reduced from S_k to $S_{k_+} \times S_{k_-}$. To see the breaking of the gauge group more explicitly, notice that the partition function for an open string with both ends on D3-branes of the same $SO(2)$ charge is given by [10]

$$\int_0^\infty dl \ \langle D3+|e^{-lH_{cl}}|D3+\rangle = \int_0^\infty \frac{dt}{2t} \ \mathrm{Tr} \ e^{-2tH_o} \frac{1+(-1)^F}{2} \frac{1+g}{2} \, , \tag{2.2}$$

where g is the inversion of $x^6, \ldots, x^9$. On the other hand, the partition function of an open string with ends on D3-branes of opposite $SO(2)$ charge is

$$\int_0^\infty dl\langle \ D3+|e^{-tH_{cl}}|D3-\rangle = \int_0^\infty \frac{dt}{2t} \ \mathrm{Tr} \ e^{-2tH_o} \frac{1+(-1)^F}{2} \frac{1-g}{2} \, . \tag{2.3}$$

In the first case, only the string modes which are invariant with respect to g are retained. In the second case, they are projected out, and instead only the antiinvariant ones are retained. Any $U(k)$ transformation which is not in $U(k_+) \times U(k_-)$ mixes the two types of open string states and is inconsistent with the orbifold projection.

Let us consider in more detail how the orbifold projection acts on the adjoint scalars living on the worldvolume of D3-branes.[1] Let α run from 3 to 5, and let i run from 7 to 9. The Higgs fields ϕ^α describe the oscillations of D3-branes in the directions parallel to the orbifold plane, while ϕ^i describe the oscillations transverse to it. The discussion in the previous paragraph implies that they satisfy

$$\begin{aligned}\phi^\alpha(x^0, x^1, x^2, -x^6) &= \Omega\, \phi^\alpha(x^0, x^1, x^2, x^6)\Omega^{-1}\,,\\ \phi^i(x^0, x^1, x^2, -x^6) &= -\Omega\, \phi^i(x^0, x^1, x^2, x^6)\Omega^{-1}\,,\end{aligned} \tag{2.4}$$

where Ω is a $k \times k$ matrix of the form

$$\Omega = \begin{pmatrix} \mathbf{1}_{k_+\times k_+} & 0 \\ 0 & -\mathbf{1}_{k_-\times k_-} \end{pmatrix}. \tag{2.5}$$

In the bulk of the D3-brane worldvolume the supersymmetry is still $N = 4$, and the gauge group is $U(k)$. Thus it is useful to think of the orbifold projection as imposing certain boundary conditions on the $N = 4$ $U(k)$ gauge theory living on the half-plane $x^6 > 0$. From the point of view of $N = 2$ SUSY the fields of this theory fall into a vector multiplet of $U(k)$ and a hypermultiplet in the adjoint of $U(k)$. ϕ^α and ϕ^i are the scalars in the vector multiplet and hypermultiplet, respectively. In view of eqs. (2.4,2.5) it is convenient to represent the fields as block matrices of the form

$$\begin{pmatrix} A_{k_+\times k_+} & B_{k_+\times k_-} \\ C_{k_-\times k_+} & D_{k_-\times k_-} \end{pmatrix}.$$

We will refer to A, D and B, C as diagonal and off-diagonal components of a field, respectively. Eq. (2.4) implies that the diagonal components of the scalars in the vector multiplet satisfy Neumann boundary conditions at $x^6 = 0$, while their off-diagonal components satisfy Dirichlet boundary conditions. For the scalars in the hypermultiplet the role of Dirichlet and Neumann conditions is reversed. By virtue of $N = 2$ SUSY this completely determines the boundary conditions for all fields at $x^6 = 0$.

Armed with this knowledge we now consider some more complicated brane configurations. In order to obtain a setup which yields a three-dimensional gauge theory at low energies, one must terminate D3-branes at some object located at $x^6 = L > 0$. One possibility is to terminate them at another orbifold 5-plane, so that the background becomes becomes $\mathbf{R}^6 \times (\mathbf{R}^3 \times \mathbf{S}^1)/\mathcal{I}$. Now every D3-brane is charged with respect to two $SO(2)$s. We will refer to the "old" $SO(2)$ as $SO(2)_1$, and call the new one $SO(2)_2$. Since $SO(2)_1$ and $SO(2)_2$ charges can be chosen idependently, a variety of possibilities arise. We will consider only two special cases.

[1]The following argument is due to A. Sen.

In the first case, all D3-branes have the same charge with respect to $SO(2)_2$. This means that at the second orbifold plane all scalars in the vector multiplet have Neumann boundary conditions, while all scalars in the hypermultiplet have Dirichlet boundary conditions. At energies much smaller than $1/L$ only the modes which have Neumann boundary conditions on both ends survive. Thus in this case the low-energy theory has only an $N = 2$ vector multiplet of $U(k_+) \times U(k_-)$. This theory has only a Coulomb branch. The flat directions correspond to the motion of D3-branes along the orbifold plane. At a generic point of the moduli space, when all D3-branes are far apart, the gauge group is broken down to $U(1)^k$. When we bring all D3-branes together, it gets enhanced to $U(k_+) \times U(k_-)$.

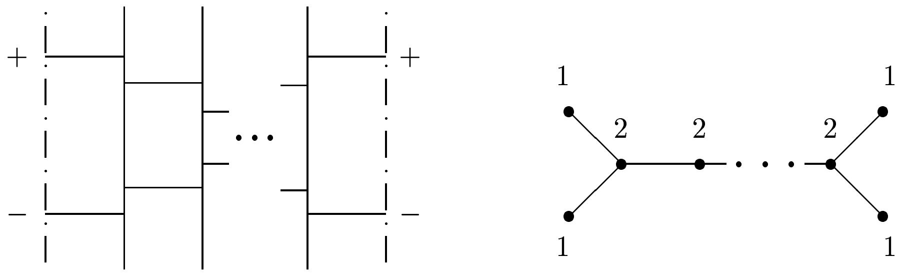
In the second case, we choose $SO(2)_2$ charges to be the same as $SO(1)_1$ charges. Similar reasoning shows that the low-energy theory contains a vector multiplet of $G = U(k_+) \times U(k_-)$ and a pair of hypermultiplets in the $(k_+, \overline{k_-})$ representation of G. The theory has both Higgs and Coulomb branches. The Higgs branch corresponds to the possibility that D3-branes of opposite $SO(2)$ charges coalesce, detach from the orbifold planes, and move off in the x^7, x^8, x^9 directions. If $k_+ \neq k_-$, complete Higgsing is impossible, so the Higgs branch is really a mixed Higgs-Coulomb branch. When we bring all D3-branes together, $U(k_+) \times U(k_-)$ is restored, and hypermultiplets become massless.

One can also terminate D3-branes on an NS5-brane parallel to the orbifold plane. It follows from the analysis of ref. [1] that the boundary conditions at the NS5-brane remove the hypermultiplet from the low-energy spectrum, so the resulting low-energy theory is the same as in the first case above.

3. Mirror symmetry for $Sp(k)$ gauge theories in $d = 3$

Consider the following brane configuration: two D3-branes stretched between two orbifold planes located at $x^6 = 0$ and $x^6 = L$, and $n - 2$ NS5-branes parallel to the orbifold plane. Let the positions of the NS5-branes in the x^7, x^8, x^9 be the same as those of the orbifold planes. We must also specify the $SO(2)$ charges of the D3-branes. We choose the two D3-branes to have opposite charges with respect to both $SO(2)_1$ and $SO(2)_2$. The resulting configuration is shown in Figure 1a. Note that D3-branes can break at the NS5-branes.

From the analysis of the previous section it follows that the low-energy theory on D3-branes is an $N = 4$ gauge theory in $d = 3$ with gauge group $U(1)^4 \times U(2)^{n-3}$. The hypermultiplets arise only from strings stretched across NS5-branes, as in ref. [1]. The gauge group and matter content can be summarized by a quiver diagram [15], which is nothing but a D_n affine Dynkin digram (figure 1). The nodes correspond to gauge group factors, and the Dynkin labels tell us the rank of the unitary group sitting at a particular node. It was argued in ref. [5] that this theory is mirror-symmetric to an $SU(2)$ gauge theory with n flavors. Now we can easily derive this



(a) (b)

Figure 1: (a) The dash-dotted lines are orbifold 5-planes. The solid vertical lines are NS5-branes. The solid horizontal lines are D3-branes. The horizontal direction corresponds to x^6, while the vertical direction corresponds to x^3, x^4, and x^5 collectively. (b) D_n quiver diagram encoding the gauge group and matter content of the low-energy theory on the branes. Each node corresponds to a unitary group, and the labels on them specify their rank. The solid lines connecting the nodes correspond to bifundamental hypermultiplets.

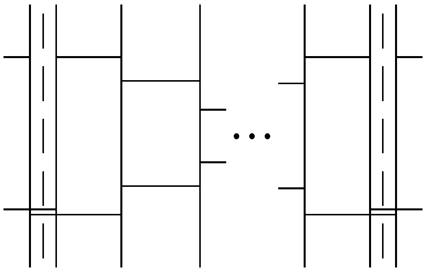
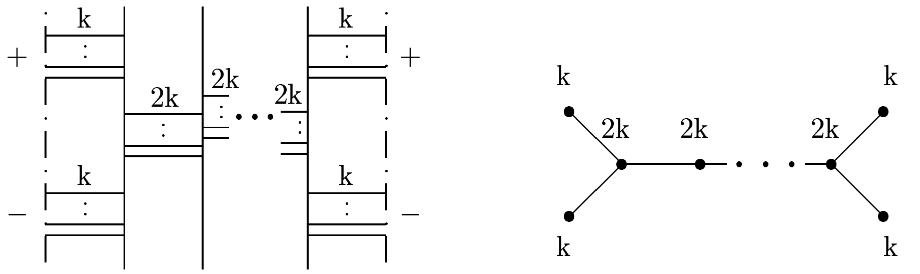


Figure 2: The S-dual of the configuration in figure 1a. The dashed lines are $O5^-$ planes, the solid vertical lines are D5-branes, the solid horizontal lines are D3-branes. For clarity we moved D5-branes slightly away from the orientifold planes. We are working on the double cover of the orientifold background, but only part of it is shown in the picture. In particular, the mirror images of only two D5-branes are shown.

statement by applying S-duality to the above brane configuration. As discussed in the introduction, the S-dual of the orbifold plane is an $O5^-$ plane with a D5-brane on top of it. NS5-branes are dual to D5-branes, and D3-branes are self-dual. We also have to figure out what the $SO(2)$ charges of D3-branes turn into under S-duality. To this end it is convenient to work on the double cover of $\mathbf{R}^6 \times (\mathbf{R}^3 \times \mathbf{S}^1)/\mathcal{I}$, and to move the D5-branes slightly away from the $O5^-$ planes. The resulting S-dual configuration is shown in figure 2. Note that the $SO(2)$ charge being positive or negative corresponds to D3-branes ending on the D5-brane or its mirror image. Now it is easy to see that the brane configuration in figure 2 yields at low energies an $SU(2)$ gauge theory with n flavors of fundamentals.

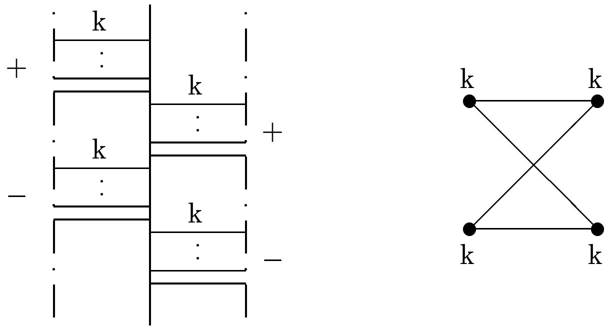


(a) (b)

Figure 3: (a) Brane construction of the mirror for $Sp(k)$ gauge theory with an antisymmetric tensor and n fundamentals. The notation is like in figure 1. (b) Quiver diagram encoding the gauge group and matter content of the brane configuration on the left.

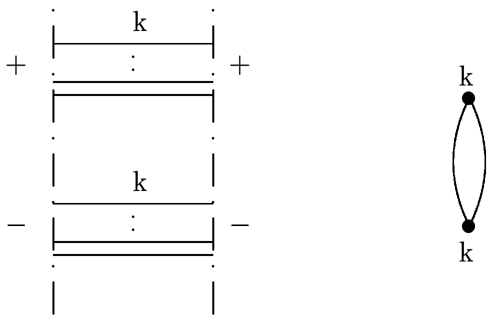
This argument can be generalized. It has been argued in ref. [6] that the mirror of $Sp(k)$ gauge theory with n fundamentals and an antisymmetric tensor is the theory defined by the quiver in figure 3b. To derive this statement from string theory we replace each D3-brane in figure 1a with k D3-branes with identical $SO(2)$ charges. The resulting configuration is shown in figure 3a. Reasoning similar to that in the previous paragraph shows that the S-dual brane configuration yields an $Sp(k)$ gauge theory with n fundamentals and an antisymmetric tensor. Thus the mirror symmetry for these two gauge theories is a consequence of S-duality of IIB string theory.

An interesting special case of this mirror symmetry occurs for $n = 2$ and 3. Formally, the mirror of $Sp(k)$ theory with an antisymmetric tensor and 2 or 3 fundamentals must be a D_2 or D_3 quiver theory, respectively. However, D_n quivers really make sense only for $n > 3$. To figure out the answer for smaller n one may start with a mirror pair for $n > 3$ and then add a large mass term for some of the fundamentals of $Sp(k)$. In the mirror theory this corresponds to adding large FI terms which Higgs the gauge group. From the point of view of brane configurations, this is achieved by moving off some of the NS5 branes in the x^7, x^8, x^9 directions, so that D3-branes reconnect. First consider integrating out all but three fundamental flavors, so that at low energies we have a theory with $n = 3$. In the brane configuration in figure 3a we move off all but one of the NS5 branes in the x^7, x^8, x^9 directions. The resulting brane configuration is shown in figure 4a. According to the rules of the previous section, this theory has gauge group $U(k)^4$ and a matter content summarized by the quiver in figure 4b. All the hypermultiplets arise from strings stretched across the NS5-brane. The quiver in figure 4b is simply a perversely drawn affine A_3 quiver. Thus $D_3 = A_3$, as one could have guessed. To get the mirror for $n = 2$ one moves off the last remaining NS5-brane, and gets the configuration in figure 5a. The gauge group is now $U(k) \times U(k)$, and the matter content is summarized by the quiver in figure 5b. It is the affine A_1 quiver, therefore $D_2 = A_1$. Note that in this case the



(a) (b)

Figure 4: (a) The brane construction of the mirror for $Sp(k)$ gauge theory with an antisymmetric tensor and 3 fundamentals. The notation is the same as in figure 1. (b) Quiver diagram encoding the gauge group and matter content of the brane configuration on the left.



(a) (b)

Figure 5: (a) The brane construction of the mirror for $Sp(k)$ gauge theory with an antisymmetric tensor and 2 fundamentals. The notation is the same as in figure 1. (b) Quiver diagram encoding the gauge group and matter content of the brane configuration on the left.

hypermultiplets come from strings stretched "vertically" between the two stacks of D3-branes. These hypermultiplets make appearance only for $n = 2$, because only in this case the boundary conditions on both ends are Neumann.

These results may seem puzzling. Indeed, a gauge theory described by an affine A_n quiver has mass deformations [5]. Its mirror must therefore have FI deformations. But $Sp(k)$ theories do not have FI terms! We suggest the following resolution. S-duality of IIB string theory seems to predict that superconformal theories which $Sp(k)$ theories in question flow to have deformations with the quantum numbers of

FI terms. However these deformations need not have a simple description in terms of the gauge theory Lagrangian in the ultraviolet. Thus we are led to the conclusion that there are "hidden FI terms" in $Sp(k)$ theories with an antisymmetric tensor and 2 or 3 fundamentals. To count the number of hidden FI deformations we simply count the number of independent mass deformations of the mirror theories. In this way one can see that for $k = 1$ there is only one hidden FI deformation, while for $k > 1$ there are two.

It is important to note that although A_3 and A_1 quiver theories have mass deformations, they are not visible in our brane realization of these theories. We believe this is related to the fact that FI deformations in the mirror gauge theories are not visible classically.

As a matter of fact, it is often the case that certain perturbations of a CFT do not have a Lagrangian description in terms of the theory in the ultraviolet. For example, the critical Ising model in d=2 is equivalent to a free Majorana fermion. Perturbation by magnetic field is described by the twist operator which is not local with respect to the fermion field. Therefore this perturbation does not have a Lagrangian description. A more closely related example is afforded by the $N = 4$ $d = 3$ $U(k)$ gauge theory with an adjoint hypermultiplet and n fundamentals. The adjoint of $U(k)$ contains a singlet and an adjoint of $SU(k)$, hence this theory has $n + 1$ mass deformations: a mass term for the singlet, a mass term for the adjoint of $SU(k)$, and $n - 1$ mass terms for the fundamentals. The mirror theory [6] has only n FI terms. A more detailed analysis reveals that it is the FI term corresponding to the mass difference of the singlet and the $SU(k)$ adjoint that is absent from the Lagrangian. We conclude that the mirror theory has a hidden FI deformation. Note again that the "missing" deformation, although present in field theory, is not visible in the brane configuration of ref. [6].

The existence of FI deformations is equivalent to the existence of conserved $U(1)$ currents [16]. Thus our findings imply that $Sp(k)$ theories with 2 or 3 flavors of fundamentals and an antisymmetric tensor have accidental $U(1)$ symmetries in the infrared.

A_1 and A_3 quiver theories are also mirror to $U(k)$ gauge theories with an adjoint and 2 and 4 flavors of fundamentals, respectively [6]. If we set $k = 1$, we come to a somewhat surprising conclusion that $U(1)$ theories with 2 and 4 electrons are equivalent in the infrared to $SU(2)$ theories with 2 and 3 fundamentals, respectively. In the remainder of this section we give some field theoretical arguments to support these equivalences.

The moduli space of the $SU(2)$ theory with 3 fundamentals consists of a Higgs branch of quaternionic dimension 3 and a Coulomb branch of quaternionic dimension 1 which meet at a single point [17]. The global $SO(6)$ flavor symmetry acts on the Higgs branch, but not on the Coulomb branch. It is also the symmetry of the interacting conformal theory at the intersection of the two branches. All these statement are also true for the $U(1)$ theory with 4 electrons, essentially by virtue of the isomorphism $SU(4) \simeq SO(6)$. The situation with the other equivalence is a bit more involved. The Higgs branch of the $SU(2)$ theory with 2 electrons consists of two disconnected pieces of quaternionic dimension 1 [18]. They intersect the Coulomb branch at two distinct points [17]. Each piece is isomorphic to $\mathbf{C}^2/\mathbf{Z}_2$, and the Coulomb branch has $\mathbf{Z}_2$ orbifold singularities at the points where it meets the Higgs branch. The flavor group is $SO(4) \simeq SU(2) \times SU(2)$. The two $SU(2)$ factors act separately on the two pieces of the Higgs branch. As a consequence, the interacting conformal theory at the intersection of the Coulomb branch and one of the Higgs branches has only $SU(2)$ global symmetry, and not $SO(4)$. On the other hand, the $U(1)$ theory with 2 electrons has a single Higgs branch isomorphic to $\mathbf{C}^2/\mathbf{Z}_2$. The Coulomb branch also has a $\mathbf{Z}_2$ singularity at the intersection point [17]. The global symmetry of the Higgs branch and the interacting conformal theory at the intersection is $SU(2)$. Thus the moduli spaces and global symmetries of the $SU(2)$ theory with 2 fundamentals and $U(1)$ theory with 2 electrons match in the neighborhoods of singular points. Note that in the strong coupling limit the two singular points on the Coulomb branch of the $SU(2)$ theory are very far apart, so one is justified in considering only the neighborhood of one of them. Thus field-theoretical analysis supports the proposed equivalences.

4. D_n quiver gauge theories in $d = 4$

In this section we use branes ending on orbifolds to construct a class of finite $N = 2$ $d = 4$ gauge theories, along the lines of ref. [2]. We explain the relationship of these theories to instantons on $\mathbf{R}^2 \times \mathbf{T}^2$ and give a simple derivation of their S-duality group. We then obtain an exact solution of these theories using the method of ref. [9].

4.1 D_n quiver theories and D_n instantons

We start with the configuration in figure 3a and T-dualize it along the x^3 direction. The only effect of this T-duality is to turn D3-branes into D4-branes. The orbifold projection remains unchanged, and IIB NS5-branes become IIA NS5-branes. The low-energy theory on the D4-branes is now a $d = 4$ $N = 2$ theory, and its gauge group and matter content can be determined using the results of section 2. One subtlety is the freezing of $U(1)$s due to the bending of NS5-branes [2]. The rule of thumb is that all $U(1)$s that are not free will freeze out. In the case at hand it implies that the gauge group is $SU(2k)^{n-3} \times SU(k)^4 \times U(1)$. The $U(1)$ vector multiplet describes the motion of all D4-branes together along x^4, x^5.

The matter content of the low-energy theory is summarized by the D_n quiver diagram (figure 3b), where each node now corresponds to a special unitary group. We will call this theory a D_n quiver theory, for short. Note that in ref. [8] the D_n

quiver theory was defined without the $U(1)$ factor. Since the $U(1)$ vector multiplet is free, it plays almost no role in our discussion.

The D_n quiver theory is finite and has interesting S-duality properties [8]. Namely, its duality group (neglecting the $U(1)$) is the fundamental group of the moduli space of flat irreducible $SO(2n)$ connections on $\mathbf{T}^2$. More generally, it was shown in ref. [8] that for a quiver to yield a finite $N = 2$ gauge theory, it has to be one of the affine A, D, E Dynkin diagrams. The duality group for an A, D, E quiver theory is the fundamental group of $\mathcal{M}_{A,D,E}$, the moduli space of irreducible flat A, D, E connections on $\mathbf{T}^2$. These results were obtained using geometric engineering.

In the case of A_n quiver theories alternative derivations are known [2, 9]. In ref. [2] A_n quiver theories (there named "elliptic models") were constructed from branes in IIA string theory. These models can be solved by lifting brane configurations to M-theory, and the duality group turns out to be the fundamental group of $\mathcal{M}_{n+1}$, the moduli space of an elliptic curve with $n+1$ marked points. One can easily see that this agrees with ref. [8]. These brane configurations can also be analyzed without resort to M-theory [9]. To this end one compactifies the brane configuration on a circle of finite radius and applies a sequence of T and S-dualities, thereby mapping the Coulomb branch of the original theory to the Higgs branch of some mirror theory. The latter does not receive quantum corrections and can be analyzed exactly. It is the last method that we will apply to D_n quiver theories.

Following ref. [9], we compactify the x^3 direction on a circle of radius R and perform T-duality along it. This direction is parallel to all branes. It is easy to see what the result is: the Neveu-Schwarz objects (NS5-branes and orbifold planes) remain unchanged, while D4-branes become D3-branes. Next we perform S-duality, so that NS5-branes become D5-branes, and orbifold 5-planes become $O5^-$ planes with D5-branes on top of them. D3-branes are self-dual. The total number of D5-branes is n, counting those which coincide with the $O5^-$ planes. Recall that in the original brane configuration we put NS5-branes on top of the fixed points of the orbifold because we wanted to have a good CFT description of the background. Now that the brane configuration contains only Ramond-Ramond objects, we can move the D5-branes off the fixed points, and still have a CFT description. This is important, since this deformation corresponds to varying the relative gauge couplings of two $SU(k)$s at the ends of the quiver – see figure 6.

At this point there are two ways to proceed. One possibility, extensively discussed in ref. [9], is to T-dualize back along x^3. This transformation turns D3-branes into D4-branes wrapped around $\mathbf{T}^2$ parametrized by x^3, x^6. $O5^-$ planes turn into $O4^-$ planes. Note that there are four $O4^-$ planes located at four points of $\mathbf{T}^2$, so D4-branes are effectively wrapped on $\mathbf{T}^2/\mathbf{Z}_2$. Finally, D5-branes turn into another set of D4-branes which we call D4$'$. D4$'$-branes are located at points of the above-mentioned $\mathbf{T}^2$. Fundamental strings stretched between D4-branes and D4$'$-branes give rise to hypermultiplets localized at points on $\mathbf{T}^2$. Thus the theory living on

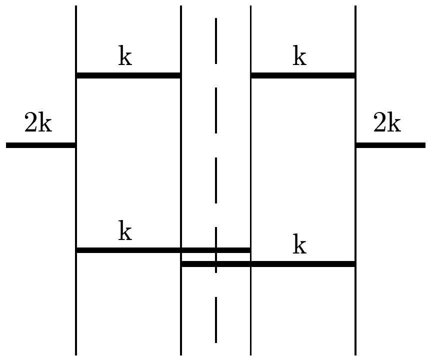


Figure 6: The dashed line is the IIA "orbifold" with negative fivebrane charge. Solid vertical lines are NS5-branes. Thick horizontal lines are stacks of D4-branes. We are working on the double cover of the orbifold background, and only the neighborhood of one fixed point is shown. When an NS5-brane and its mirror image coincide with the fixed point, the gauge couplings of the two $SU(k)$s at the end of the quiver are the same.

D4-branes is an impurity theory [19, 20, 21]. Its supersymmetric ground states correspond to solutions of certain first-order differential equations on $\mathbf{T}^2$ [21]. These equations are Hitchin equations with sources at the locations of the impurities. The sources are rank one matrices, and their residues correspond to the masses of the bifundamentals in the original theory [9]. The effect of the orientifolds is to impose a certain projection on the connection and Higgs field appearing in the Hitchin equations [21]. The moduli space of Hitchin equations is the Coulomb branch of the D_n quiver theory we started from. In the next subsection we will use this to find the Seiberg-Witten curve for these theories.

The second possibility is to T-dualize along x^6. $O5^-$ planes and D5-branes become an $O6^-$ plane and n D6-branes wrapped on the $\mathbf{T}^2$ parametrized by x^3, x^6. This $\mathbf{T}^2$ is dual to the $\mathbf{T}^2$ on which the Hitchin equations live. As for D3-branes, they become D2-branes parallel to D6-branes and localized in x^3, x^6. The gauge theory on the D6-branes is an $SO(2n)$ theory with sixteen supercharges. D2-branes can be viewed as BPS instantons of this theory. More precisely, they are instantons on $\mathbf{R}^2 \times \mathbf{T}^2$, where $\mathbf{R}^2$ has coordinates x^4, x^5. Thus the Coulomb branch of the D_n quiver theory on a circle is the same as the moduli space of $SO(2n)$ instantons on $\mathbf{R}^2 \times \mathbf{T}^2$. To specify the moduli space of such instantons completely, one needs to fix their behaviour at infinity, where they become flat connections on $\mathbf{T}^2$ [21]. The moduli space of flat connections being nontrivial, the Coulomb branch depends on the corresponding moduli. These moduli are nothing but the gauge couplings and theta angles of the D_n quiver theory. To see this note that Hitchin equations are the Nahm transform of instanton equations on $\mathbf{R}^2 \times \mathbf{T}^2$, and the positions of impurities encode the asymptotic behaviour of the instantons [21]. On the other hand, tracing the dualities one can easily see that the positions of impurities on $\mathbf{T}^2/\mathbf{Z}_2$ are determined by the positions of NS5-branes in the brane configuration in figure 3a. Therefore these positions encode the gauge couplings of the quiver theory.

To summarize, the moduli space of $2k$ $SO(2n)$ instantons on $\mathbf{R}^2\times\mathbf{T}^2$ asymptotic to a fixed flat connection is the Coulomb branch of the compactified D_n quiver theory, with complexified gauge couplings determined by the flat connection. It follows that the S-duality group of the D_n quiver theory is the fundamental group of the moduli space of flat irreducible $SO(2n)$ connections on $\mathbf{T}^2$. This agrees with the result of ref. [8]. (The irreducibility condition arises because reducible flat connections correspond to some of the gauge couplings being infinite, as one can easily convince oneself.) A_n quiver theories are related to moduli spaces of $U(n)$ instantons on $\mathbf{R}^2 \times \mathbf{T}^2$ in a similar manner [9]. Their duality group is therefore given by the π_1 of the moduli space of flat irreducible $U(n)$ connections on $\mathbf{T}^2$, in agreement with refs. [2, 8].

4.2 Solution of the models

As explained in the previous subsection, the moduli space of the D_n quiver theory compactified on a circle is identical to the moduli space of solutions of Hitchin equations on a punctured $\mathbf{T}^2$ satisfying a certain projection condition. This projection is the orientifold projection which acts on the coordinates of $\mathbf{T}^2$ by reflection. Thus it has four fixed points.[2] $\mathbf{T}^2$ with its complex structure is an elliptic curve which we call Σ. Let z be a "flat" holomorphic coordinate on Σ. The orientifold projection acts by $z \to -z$. The action on the connection and the Higgs field ϕ is of the form

$$A_z(-z) = MA_z(z)^T M^{-1}\,, \quad \phi(-z) = M\phi(z)^T M^{-1}\,, \tag{4.1}$$

for some constant matrix M. To figure out M one recalls that at the location of the $O4^-$ plane the gauge group on D4-branes reduces from $U(2k)$ to $Sp(k)$. This means that M is a nondegenerate antisymmetric matrix. Without loss of generality we can set it to

$$M = i\sigma_2 \otimes \mathbf{1}_{k\times k}\,. \tag{4.2}$$

Let us sketch an alternative way of deriving eq. (4.2) from the properties of Nahm transform for $SO(2n)$ instantons. For simplicity, let us start with BPS-saturated $SO(2n)$ *monopoles*. Any $SO(2n)$ monopole can be embedded into $U(2n)$, therefore Nahm data for $SO(2n)$ monopoles are a subset of Nahm data for $U(2n)$ monopoles. This subset is easy to characterize [22]: it is a set of all $U(2n)$ Nahm data which satisfy the condition eq. (4.1) with M as in eq. (4.2), where z is now a real parameter

[2] Alternatively, the moduli space in question can be regarded as the moduli space of *all* solutions of Hitchin equations on an orbifold Riemann surface $\mathbf{T}^2/\mathbf{Z}_2$.

on which the Nahm matrices depend. It is trivial to generalize this to instantons on $\mathbf{R}^3 \times \mathbf{S}^1$, and even to instantons on $\mathbf{R}^2 \times \mathbf{T}^2$.

We thus have an implicit description of the hyperkähler metric on the Coulomb branch of the compactified D_n quiver theory. In the decompactification limit (i.e. when the radius of the compact circle is taken to infinity) only a distinguished complex structure of this manifold is important [9]. This complex structure is encoded in the spectral cover C defined by the equation

$$\det(v - \phi(z)) = 0 \,. \tag{4.3}$$

This says that C is a $2k$-fold cover of Σ. The importance of C lies is in the fact that it is the Seiberg-Witten curve for the gauge theory we set out to solve [23, 9]. In the remainder of this paper we provide an explicit description of C. As a first step, we rewrite eq. (4.3) in a more explicit form:

$$v^{2k} + f_1(z)v^{2k-1} + \cdots + f_{2k}(z) = 0 \,. \tag{4.4}$$

As in ref. [9], Hitchin equations imply that $f_1, \ldots, f_{2k}$ are meromorphic functions on Σ with simple poles at the $2n$ punctures. The condition eq. (4.1) further implies that $f_1, \ldots, f_{2k}$ are even elliptic functions. More concretely, let Σ be given by

$$y^2 = (x - e_1)(x - e_2)(x - e_3) \,. \tag{4.5}$$

An even elliptic function with $2n$ simple poles has the form

$$\sum_{i=1}^{n} \frac{a_i}{x - x_i} + b \,. \tag{4.6}$$

The poles are the solutions of $\mathcal{P}(z) = x_i, i = 1, \ldots, n$, where $\mathcal{P}(z)$ is the Weierstrass elliptic function. We see that the space of even elliptic functions with $2n$ simple poles has dimension $n + 1$. Therefore there are $2k(n + 1)$ parameters in the curve eq. (4.4). However, there is an additional constraint following from eqs. (4.1,4.2): all the eigenvalues of $\phi(z)$ are doubly degenerate at the four fixed points of the $\mathbf{Z}_2$ action $z \to -z$. Therefore all roots of eq. (4.4) are doubly degenerate for $x = e_1, e_2, e_3, \infty$. This imposes $4k$ extra constraints on the coefficients of eq. (4.4), thereby reducing the number of parameters to $2k(n - 1)$. These parameters are interpreted as the moduli of the Coulomb branch and the hypermultiplet masses. As in ref. [2], the asymptotic behaviour of C is encoded in the residues of f_1, which are therefore interpreted as the hypermultiplet masses. Thus we have n hypermultiplet masses and $2k(n-1)-n$ moduli. This agrees with the expected number of mass deformations and moduli in the $SU(2k)^{n-3} \times SU(k)^4 \times U(1)$ theory.

A final remark concerns an extension of our results to E_n quiver theories of ref. [8]. In view of the relation between A_n and D_n quiver theories and A_n and

D_n instantons, it is natural to expect that the exact solution of E_n quiver theories is described by a Hitchin system which is a Nahm transform of E_n instantons on $\mathbf{R}^2 \times \mathbf{T}^2$. Unfortunately, we do not know what the Nahm transform looks like for exceptional groups. Neither is it clear what sort of brane configuration could yield an exceptional quiver. We believe that a solution of the former problem could provide a key to the solution of the latter.

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